J. Fluid Mech. (1975), *vol.* 71, *part* 4, *pp.* 689–700
Printed in Great Britain

Sound radiation into uniformly flowing fluid by compact surface vibration

By J. E. FFOWCS WILLIAMS
Engineering Department, University of Cambridge

AND D. J. LOVELY
Department of Mathematics, Imperial College, London

(Received 7 February 1975)

This paper describes a model problem where compact surface vibration radiates sound into a subsonically flowing fluid. There are two distinct acoustic effects. First, the radiation is increased by flow by an amount proportional to $5M^2$ and that increase is shown by a general argument to arise from an enhanced surface damping and work done by the flow to overcome drag in the ratio 2 : 1. Second, the acoustic source strength is affected and resonance frequencies are significantly modified by flow. The main effect is that flow induces on the surface a force proportional to the displacement which opposes the action of natural surface elasticity. A critical velocity exists beyond which the surface is unstable; the stability limit is determined. The surface motion might be regarded as an acoustic monopole, but since aerodynamic fields are determined by the *rate* of change of the *rate* of mass outflow, the frequency dependence is more that of a quadrupole. Convective amplification of the sound is also shown to be that characteristic of quadrupole sources. This result indicates that real simple fields may be more sensitive to convection than might be expected from past studies of simple inhomogeneities satisfying a convected wave equation.

1. Introduction

The influence of flow on the level and directionality of sound generated by internal sources is known to be sometimes significant and often perplexing. Mani (1974) has described important changes in the fields of idealized mathematical models of sources immersed in a flow and Ffowcs Williams (1974) has shown how the proximity of flow dramatically influences sound generated by turbulence. The noise radiated by an aircraft in flight does not seem to be predictable from a knowledge of the engine noise under equivalent static conditions. Hoch & Hawkins (1974) gave evidence that jet noise actually increases in some directions despite the alleviating effect of the flight reduction in relative jet velocity. Of course, source motion is known to influence the radiated sound but these recent results are of a degree that is not easily anticipated from the numerous moving-source studies available in the literature.

Lowson's (1965) comprehensive analysis of the sound generated by singularities in motion allows a simple evaluation of convective effects on compact sources of any known type, while the general results describing aerodynamic sound from turbulence and surfaces in arbitrary motion (Ffowcs Williams & Hawkings 1969) show that convection of any single source element is simply accounted for in the various effects attributable to the Doppler contraction of the wave scale. To those effects may have to be added any influence that motion may have on the relative strengths of different sources, which, if substantial, would make the field of isolated model sources less relevant in practice. In that event, the exact expressions worked out by Ffowcs Williams & Hawkings, which are actually integral forms of the governing wave equation, could not be regarded as effectively displaying all the important convective effects; some would still reside in an implicit interdependence of the various source elements, the degree of which might vary with changes in the convection, or flow, velocity. This actually seems to be the case, and we illustrate the point in this paper through a detailed study of what is possibly the simplest source type.

We describe the effect of uniform flow on the sound generated by a vibrating piston set in an infinite otherwise plane baffle. This problem would at first sight appear to be well modelled by an acoustic monopole, whose (linear) field is increased by source motion by the second power of the Doppler factor; cf. Ffowcs Williams & Hawkings (1969, equation 5.1). In fact, we show that the obvious interpretation of that integral formula is wrong and that, when carefully analysed, it contains another power of the Doppler factor making the field of this apparent simple monopole increase in intensity according to $(1 - M\cos\theta)^{-6}$, the dependence thought typical of quadrupoles! The effect is, in this case, correctly anticipated when it is appreciated that the monopole, in displacing fluid, also displaces mean-flow momentum, which induces a significant dipole, phase locked to the monopole; it is also correctly anticipated when it is appreciated that the fundamental aeroacoustic source strength is the *rate* of change of the *rate* of mass outflow, and that therefore the field is modified by the two Doppler factors characteristic of quadrupoles. But we have reason to believe this simple explanation to be adequate only when the flow is weakly perturbed from its mean state. One of our objectives in presenting this analysis of a simple boundary-value problem is to help to clarify the flow effects on model sources and to bring to light the fact that general formulae which display convective effects sometimes need cautious handling.

The problem has further significance in that it throws light on the acoustic interaction with flow-induced instabilities and illustrates how flow over compliant surfaces can feed energy into the acoustic field. Surfaces in contact with flow often vibrate in practice, sometimes, as in the SONAR transmitter, because they are deliberately forced and sometimes because they are made compliant and damped to respond to and absorb acoustic energy, as is the case in acoustically treated aircraft engine intakes and exhausts. Flow over a compliant surface is often unstable, and high damping cannot always inhibit the instabilities (Benjamin 1963). Are these instabilities coupled to the sound and, in an acoustic liner, could they possibly produce more sound than the liner absorbs? It was observed by Dean (1972) and Tester (1973) that some acoustic modes grow in the presence of flowing fluid. And even when the motion is stable, convective amplification of sound results in an enhancement of any acoustic field leaving a vibrating surface. What is the origin of the extra energy in that field, and does flow substantially change the mode in which surfaces resound? This issue is probably too complex readily to admit general conclusions and progress must rest on detailed observation in experiments and on precise analysis of the tractable model problems. The problem treated here is such a one, and from its solution we determine that instability is a real issue and that flow effects can radically change the vibrational response of compliant surfaces and feed energy directly into the sound as well as causing the surface vibration to convert its energy into sound more effectively.

We assume in our model that the flow velocity is constant and that the surface vibration has no bearing on the overall mean state. The fluid is inviscid. Boundary-layer effects are obviously important in practice but are too difficult for us to treat analytically. Consequently our model will have limited practical application but should be relevant whenever the boundary-layer thickness is much smaller than the characteristic scale of the surface vibrations. Since this problem can be solved exactly the model forms a useful base from which we can gain an understanding of the mechanics underlying the more complex situation.

Our problem can be sensibly divided into two sections. There is the purely acoustic effect of a compact source moving uniformly at low Mach number relative to a fluid. We treat this without having to specify the source details assuming that the surface displacement is independent of motion. Motion is known to increase the radiated energy by an amount proportional to M^2 (Lighthill 1952). In fact, Lighthill's analysis carries over directly since the surface source term is subject to quadrupole-like Doppler amplification, two Doppler factors arising from the two rate-of-change operators and one from the usual convective expansion of the source volume. We show in our problem that the energy increase is $5M^2$ times the energy radiated in static fluid. This energy is provided by two distinct effects. First, there is increased mechanical damping, and second, energy is extracted from the mean flow in overcoming the drag on the vibrating surface. These increases contribute energy in the ratio 2:1. The flow has also a non-acoustic effect on the surface vibration; the response amplitude, the resonance frequency and the effective mechanical damping are all affected, and this in turn alters the acoustic source strength. This is usually more important than the purely acoustic effect.

We consider a circular piston which is free to vibrate normally in an otherwise rigid plane wall but even this local problem is not trivial. There are easily foreseen difficulties associated with the singularities of the potential flow at sharp edges of the piston and baffle. Batchelor (1967, p. 226) shows that the resulting force is logarithmically singular. He goes on to say that "what we learn... is that the total force depends on the precise shape of the two boundaries close to their intersection". We emphasize this point by considering 'pistons' which do not have profile discontinuities and draw conclusions about the effect of the degree of edge curvature on the magnitude of this force. The actual magnitude is difficult to find, but we expect it to be governed by a length scale on which our potential modelling fails, possibly a boundary-layer scale or the finite curvature of practical 'sharp' edges.

The various flow effects are quantified in this paper and are shown to be quite substantial. The greatest occurs when the flow brings the piston close to resonance (or detunes a resonating piston), and we show also that the flow can lead to an instability. We deal only with the stable case but determine the limiting value of the flow velocity for which the work is valid. That limit is reached when the flow-induced suction force on a protruding piston exactly balances the stiffness of the restoring spring.

2. Sound radiated into moving flow by compact surface vibration

We consider homogeneous inviscid fluid in uniform motion parallel to a plane boundary S. The fluid occupies the upper half-space and moves with velocity $c\mathbf{M}$, c being the speed of sound. A compact section of the boundary vibrates, radiating sound to infinity. We choose a co-ordinate system $\mathbf{X}$ fixed in the fluid and write $p(\mathbf{X},t)$ for the sound pressure far away from the vibrating part of the surface:

$$\begin{aligned} p(\mathbf{X},t) &= \frac{\rho_0}{2\pi|\mathbf{X}|}\int_S \frac{\partial^2\xi}{\partial t^2}\left(\mathbf{Y}, t-\frac{|\mathbf{X}-\mathbf{Y}|}{c}\right) d^2\mathbf{Y} \\ &= \frac{\rho_0}{2\pi|\mathbf{X}|}\int_S\int_\tau \xi(\mathbf{Y},\tau)\,\delta''\left(\tau-t+\frac{|\mathbf{X}-\mathbf{Y}|}{c}\right) d\tau\, d^2\mathbf{Y}. \end{aligned} \qquad (1)$$

δ is Dirac's delta function, primes denote differentiation with respect to the argument, ρ_0 is the density of the undisturbed fluid and ξ the small displacement of the surface from its mean position. ξ is finite only within a compact region moving relative to this co-ordinate system with velocity $-c\mathbf{M}$. We therefore choose a second co-ordinate system $\mathbf{x}$, whose origin moves with the centre of the vibrating region, and write $\eta(\mathbf{x},\tau)$ for the surface elevation at time τ:

$$\xi(\mathbf{X},\tau) = \eta(\mathbf{x},\tau), \quad \mathbf{x} = \mathbf{X}+c\mathbf{M}\tau. \qquad (2)$$

The Jacobian of this transformation is unity, so that

$$p(\mathbf{X},t) = \frac{\rho_0}{2\pi|\mathbf{X}|}\int_S\int_\tau \eta(\mathbf{y},\tau)\,\delta''\left(\tau-t+\frac{|\mathbf{X}-\mathbf{Y}|}{c}\right) d\tau\, d^2\mathbf{y}. \qquad (3)$$

This equation is integrated by parts with respect to τ, and we use the fact that

$$\frac{\partial}{\partial\tau|_{\mathbf{y}\,\mathrm{const}}}\left(\tau-t+\frac{|\mathbf{X}-\mathbf{Y}|}{c}\right) = 1+\mathbf{M}.\frac{(\mathbf{X}-\mathbf{Y})}{|\mathbf{X}-\mathbf{Y}|} = 1+\frac{\mathbf{M}.\mathbf{X}}{|\mathbf{X}|} = 1+M\cos\theta, \quad \text{say},$$

from which it follows that

$$p(\mathbf{X},t) = \frac{\rho_0}{2\pi|\mathbf{X}|\,|1+M\cos\theta|^3}\int_S \frac{\partial^2\eta}{\partial\tau^2}\left(\mathbf{y}, t-\frac{|\mathbf{X}-\mathbf{Y}|}{c}\right) d^2\mathbf{y}. \qquad (4)$$

We now restrict our attention to very compact surface sources, so that the small

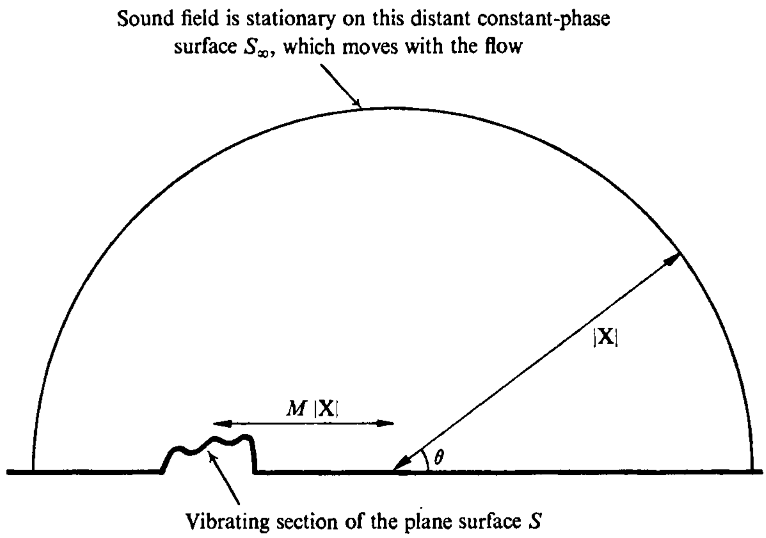

FIGURE 1. Diagram illustrating the co-ordinate system and the model problem.

differences in the retarded time are negligible; the distant wave field is then determined by the instantaneous source strength Q;

$$Q\left(t-\frac{|\mathbf{X}|}{c}\right) = \int_S \frac{\partial^2\eta}{\partial\tau^2}\left(\mathbf{y}, t-\frac{|\mathbf{X}-\mathbf{Y}|}{c}\right) d^2\mathbf{y} \tag{5}$$

at a level

$$p(\mathbf{X},t) = \frac{\rho_0 Q(t-|\mathbf{X}|/c)}{2\pi|\mathbf{X}|\,|1+M\cos\theta|^3} \tag{6}$$

and has mean-square amplitude

$$\overline{p^2}(\mathbf{X},t) = \frac{\rho_0^2\overline{Q^2}(t-|\mathbf{X}|/c)}{4\pi^2|\mathbf{X}|^2|1+M\cos\theta|^6}, \tag{7}$$

where an overbar denotes an average taken over many 'cycles'. This sixth-power dependence on the Doppler factor is precisely that obtained by Lighthill for moving quadrupoles; in fact, the detailed structure of (4) and (10) is exactly that of the Lighthill theory.

We now examine conditions on a distant control hemisphere S_∞ that lies parallel to the phase fronts radiated by the vibrating surface. Figure 1 illustrates the point that the centre of the hemisphere, of radius $|\mathbf{X}|$, has drifted with the fluid a distance $M|\mathbf{X}|$ downstream of the vibrating section while sound travelled out from its source to reach the surface. On this surface the radiation is as statistically stationary in time as the vibration which produced it.

Sound energy crosses S_∞ at a mean rate

$$\frac{\overline{p^2}}{\rho_0 c}+\frac{\mathbf{M}.\mathbf{X}\overline{p^2}}{|\mathbf{X}|\rho_0 c} \quad \text{per unit area.} \tag{8}$$

The first term represents the rate of working of the sound pressure and the second the rate at which the mean flow convects the energized fluid across S_∞. The mean radiation energy density is $\overline{p_0^2}/\rho_0 c^2$ (the sum of equal kinetic and potential parts) and the volume flux out of S_∞ is $c\mathbf{M}.\mathbf{X}/|\mathbf{X}|$, or $cM\cos\theta$, per unit area.

The total acoustic power P radiated into the moving stream is therefore

$$P = \int_{S_\infty} \frac{\overline{p^2}}{\rho_0 c}(1 + M\cos\theta)\, dS_\infty \tag{9}$$

and provided that the source vibration is statistically stationary, so that

$$\overline{p^2}(\mathbf{X}, t) = \overline{p^2}(\mathbf{x} - c\mathbf{M}t, t) = \overline{p^2}(\mathbf{x}, 0) = \overline{p^2}(\mathbf{X}, 0),$$

$$P = \int_0^\pi \frac{\rho_0^2 \overline{Q^2}}{4\pi^2|\mathbf{X}|^2} \frac{(1 + M\cos\theta)^{-5}}{\rho_0 c} \pi|\mathbf{X}|^2 \sin\theta\, d\theta = \frac{\rho_0 \overline{Q^2}}{2\pi c}\{1 + 5M^2 + O(M^4)\}. \tag{10}$$

Low Mach number flow evidently increases the sound energy radiated from a vibrating surface, the increase being $5M^2$ times the power radiated into fluid at rest.

Again this convective amplification of energy is in exact parallel with the aerodynamic sound problem; the factor $(1 - M\cos\theta)^{-6}$ for individual quadrupoles is subject to the Ffowcs Williams (1963) modification to account for the fixed source volume, only a fraction $1 - M\cos\theta$ of which is an effective sound producer.

Where does this extra energy come from? There are two potential sources, as can be seen by considering the rate at which energy crosses the source boundary surface S. In fluid-fixed co-ordinates, that energy flux is

$$P = \int_S \overline{P\,(\mathbf{X}, t) \frac{\partial \xi}{\partial t}(\mathbf{X}, t)}\, d^2\mathbf{X}, \tag{11}$$

an expression which may be written in source-fixed co-ordinates as

$$P = \int_S \overline{p\frac{\partial \eta}{\partial t}}\, d^2\mathbf{x} + cM_i \int_S \overline{p\frac{\partial \eta}{\partial x_i}}\, d^2\mathbf{x}, \tag{12}$$

because $$\frac{\partial \xi}{\partial t} = \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x_i}\left(\frac{\partial x_i}{\partial t}\right)_{\mathbf{X}} = \frac{\partial \eta}{\partial t} + cM_i \frac{\partial \eta}{\partial x_i}.$$

The first term represents the power extracted from the surface vibration, P_v say, and the second power extracted from the mean flow to overcome the steady drag force on the boundary surface, $\overline{p\,\partial\eta/\partial x_i}$ being the component of the surface stress in the i direction. We denote this drag-induced power by P_d, and evaluate it by determining the steady drag on the surface S_∞, which must equal the boundary drag on S. The distant pressure, with its zero mean value, cannot contribute to the drag on S_∞, which is comprised entirely of terms arising from the mass flux times the unsteady part of the momentum per unit mass (the steady part amounting to zero by mass conservation). The drag on S_∞ is therefore

$$D = -\int_{S_\infty} \overline{\rho u_n \frac{p}{\rho_0 c}} \cos\theta\, d^2\mathbf{x}$$

$$= -\int_{S_\infty} \overline{\left\{\rho_0 \frac{p}{\rho_0 c} + \frac{p}{c^2} cM\cos\theta\right\} \frac{p}{\rho_0 c}} \cos\theta\, d^2\mathbf{x}$$

$$= -\int_{S_\infty} \frac{\overline{p^2}}{\rho_0 c^2}(1+M\cos\theta)\cos\theta\, d^2\mathbf{x}$$

$$= -\int_0^\pi \frac{\rho_0 \overline{Q^2}}{4\pi c^2}(1+M\cos\theta)^{-5}\cos\theta\sin\theta\, d\theta$$

$$= \tfrac{5}{6}M\,(\rho_0\overline{Q^2}/\pi c^2)\,(1+O(M^2)). \tag{13}$$

The power absorbed from the mean flow to overcome this drag is

$$P_d = cMD = \tfrac{5}{6}M^2\rho_0\overline{Q^2}/\pi c. \tag{14}$$

By subtraction, since, from (10), $P = P_v + P_d = (1+5M^2)\rho_0\overline{Q^2}/2\pi c$,

$$P_v = (1+\tfrac{10}{3}M^2)\rho_0\overline{Q^2}/2\pi c. \tag{15}$$

Evidently, the increased radiation induced by the mean flow draws its energy from both the surface and the mean flow, the two parts being in the ratio 2 : 1. The radiation damping of the surface motion will thus be increased by the flow to a value $1+\frac{10}{3}M^2$ times its value in stagnant fluid. This will now be confirmed by a local calculation.

3. The local interaction between a flow and a compact vibrating surface

We continue to consider homogeneous inviscid fluid in uniform motion parallel to an unsteady plane boundary. The fluid occupies the upper half-space and moves with velocity $c\mathbf{M}$, c being the speed of sound. A compact circular section of the boundary vibrates, and we refer to this section as the piston. The remainder of the boundary, which is rigid, is the baffle. The piston is held in place by a spring whose undisturbed length keeps the piston face flush with the baffle.

To investigate the local effect of the flow, we first calculate the force exerted on the piston owing to its own small oscillations. We choose a co-ordinate system $\mathbf{X}$ fixed in the fluid and write again the pressure at $\mathbf{X}$ as

$$p(\mathbf{X},t) = \frac{\rho_0}{2\pi}\int_{S(\mathbf{Y},t)} \frac{\partial^2\xi}{\partial t^2}\left(\mathbf{Y}, t-\frac{|\mathbf{X}-\mathbf{Y}|}{c}\right)\frac{d^2\mathbf{Y}}{|\mathbf{X}-\mathbf{Y}|}.$$

ξ is non-zero only on the piston face, which is moving relative to this co-ordinate system with velocity $-c\mathbf{M}$. The force on the piston is

$$F(t) = \int_{S(\mathbf{X},t)} p(\mathbf{X},t)\, d^2\mathbf{X}. \tag{16}$$

We evaluate the integrals by expanding the retarded time in a Taylor series about the point $\mathbf{X}=\mathbf{Y}$, and this procedure is valid provided that the piston is compact, i.e. provided that

$$\left|\frac{|\mathbf{X}-\mathbf{Y}|}{c}\frac{\partial}{\partial t}(\ln\xi)\right| \ll 1. \tag{17}$$

The integrals are now transferred to the piston-fixed co-ordinate system $\mathbf{x}$,

$$x_i = X_i + cMt\delta_{1i}, \quad \xi(\mathbf{Y},t) = \eta(\mathbf{y},t), \tag{18}$$

in which the force is expressed as

$$F(t) = \frac{\rho_0}{2\pi}\int_{S(\mathbf{x})}\int_{S(\mathbf{y})}\frac{1}{|\mathbf{x}-\mathbf{y}|}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial y_1}\right)^2\eta(\mathbf{y},t)\,d^2\mathbf{y}\,d^2\mathbf{x}$$
$$+\frac{\rho_0}{2\pi}\int_{S(\mathbf{x})}\int_{S(\mathbf{y})}\sum_{j=1}^{\infty}\frac{(-1)^j|\mathbf{x}-\mathbf{y}|^{j-1}}{c^j j!}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial y_1}\right)^{j+2}\eta(\mathbf{y},t)\,d^2\mathbf{y}\,d^2\mathbf{x}. \tag{19}$$

The terms which are independent of c (i.e. the terms remaining non-zero on letting $c\to\infty$) are the 'incompressible' flow terms:

$$\frac{\rho_0}{2\pi}\int_{S(\mathbf{x})}\int_{S(\mathbf{y})}\frac{1}{|\mathbf{x}-\mathbf{y}|}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial y_1}\right)^2\eta(\mathbf{y},t)\,d^2\mathbf{y}\,d^2\mathbf{x}. \tag{20}$$

The $\mathbf{x}$ integration can be performed immediately:

$$\int_{S(\mathbf{x})}\frac{d^2\mathbf{x}}{|\mathbf{x}-\mathbf{y}|} = 4aE\left(\frac{|\mathbf{y}|}{a}\right),$$

where a is the radius of the piston. $E(k)$ is the elliptic integral of the second kind. This result is obtained by the use of Copson's (1947) formula, which states that

$$\int_0^{2\pi}\frac{d\theta}{(\rho^2+r^2-2\rho r\cos\theta)^{\frac{1}{2}}} = 4\int_0^{\min(\rho,r)}\frac{dt}{(\rho^2-t^2)^{\frac{1}{2}}(r^2-t^2)^{\frac{1}{2}}}. \tag{21}$$

We do not expect to be able to find the exact value of the force in the case of a piston whose angle of contact is $\frac{1}{2}\pi$. In that case we can write

$$\eta(\mathbf{y},t) = H(a-|\mathbf{y}|)\,\eta(t)$$

and find that the expression for the 'incompressible' force has a singularity arising from the integral

$$\frac{2\rho_0 aU^2\eta(t)}{\pi}\int_{S(\mathbf{y})}\left\{\frac{\partial^2}{\partial y_1^2}H(a-|\mathbf{y}|)\right\}E\left(\frac{|\mathbf{y}|}{a}\right)d^2\mathbf{y}. \tag{22}$$

When we integrate by parts once we obtain

$$\int\frac{y_1^2}{|\mathbf{y}|^2}\delta(a-|\mathbf{y}|)\frac{\partial}{\partial|\mathbf{y}|}E\left(\frac{|\mathbf{y}|}{a}\right)dy_1\,dy_2, \tag{23}$$

which contains a term $E'(k)$ as $k\to 1$. The singularity is negative and logarithmic.

This singularity has arisen through the inability of our linearized potential model to deal with the surface discontinuity at the piston edge. This point can be demonstrated by the following examples. Suppose that the 'piston' is flexible and is flush with the baffle at its edge, rising to a peak at its centre. During an oscillation the peak varies from $+|\eta|$ to $-|\eta|$, but the edge remains flush. The elevation of the 'piston' can be represented, for example, by the functions

$$(1-|\mathbf{y}|^2/a^2)\,H(a-|\mathbf{y}|)\,\eta(t), \quad (1-|\mathbf{y}|^2/a^2)^2 H(a-|\mathbf{y}|)\,\eta(t). \tag{24}$$

These 'pistons' lead to mean-flow-induced forces of

$$-\tfrac{4}{3}\rho a U^2 \eta(t), \quad -\tfrac{32}{45}\rho a U^2 \eta(t) \tag{25}$$

respectively. It is evident that this piston force term is a mean-flow effect. It represents a suction when the piston protrudes from the boundary and is caused by the depression when the flow accelerates past the blockage presented by the piston. The magnitude of the force is highly sensitive to any variation in the piston geometry.

One way of admitting the nonlinearity of the problem leads one to evaluate the force on the actual surface of the piston, i.e. at a height η above the baffle surface. The integral required is then

$$\frac{\rho U^2 \eta}{2\pi} \int_{S(\mathbf{x})} \int_{S(\mathbf{y})} \left\{ \frac{\partial^2}{\partial y_1^2} H(a - |\mathbf{y}|) \right\} \frac{d^2\mathbf{y}\, d^2\mathbf{x}}{\{|\mathbf{x}-\mathbf{y}|^2 + \eta^2\}^{\frac{1}{2}}}, \tag{26}$$

and this limits the previous singularity to a value

$$\rho a U^2 \eta \ln \{|\eta|/a\} + O(\eta), \quad |\eta| \ll a. \tag{27}$$

This is actually the exact expression for the force on the face of a circular piston protruding from a baffle into otherwise uniform potential flow. We cannot hope to get a meaningful value for this nonlinear mean-flow suction, as it must in any practical flow depend on boundary-layer and viscous effects. But the actual magnitude of this force is of little concern; it is bound to be expressible in the form

$$\rho a U^2 \eta \ln (\epsilon/a), \tag{28}$$

where ϵ is the length scale on which our potential modelling has broken down. The presence of the logarithmic function ensures that although our estimate of ϵ/a may not be very accurate, the error will have little effect on the value of the force. Accordingly we write the suction force as

$$\rho a U^2 \eta(t) A_0, \tag{29}$$

where A_0 is a constant.

The remaining terms in (19) can be evaluated to any desired degree of accuracy. The majority pose no difficulty but those terms of the form

$$M^{2n}|\mathbf{x}-\mathbf{y}|^{2n-1}\, \partial^{2n}[H(a-|\mathbf{y}|)]/\partial y_1^{2n}$$

have singularities of the foregoing type and must be treated in a similar way. Each term is at least order M^2 smaller than the first, and we ignore them in this low Mach number problem to obtain

$$F(t) = \tfrac{8}{3}\rho a^3 \eta''(t) - \rho_0 a U^2 \eta(t) A_0 - \frac{\rho_0 a^4 \pi}{2c} \eta'''(t) + \frac{32\rho_0 a^5 \eta^{\mathrm{iv}}(t)}{45c^2} + 4\rho_0 a^3 \eta''(t) M^2 - \frac{\rho_0 a^6 \pi \eta^{\mathrm{v}}(t)}{12c^3} - \frac{5\rho_0 a^4 \pi \eta'''(t) M^2}{3c} + O(c^{-4}). \tag{30}$$

This equation determines the flow-induced relation between the piston displacement and the surface restoring force. Any mechanical system will have another relation between the piston force and displacement. If both are ever in agreement, then a self-sustained piston oscillation is possible. When no such flutter condition exists (30) provides an expression for the impedance imposed on the piston surface by the presence of the moving compressible fluid. We examine this in detail below, an examination that is much simplified if the piston is supposed to be in harmonic motion at angular frequency ω.

4. The influence of the flow on the apparent mechanical properties of the piston

We now suppose that the harmonically ($\exp\{-i\omega t\}$) vibrating piston has mass m, is restrained by a spring of stiffness K and has a damping factor β. We accordingly define the piston impedance to be

$$Z(\omega) = -\beta + (i/\omega)(m\omega^2 - K). \tag{31}$$

The piston presents a resistance Z to any system driving it at unit velocity.

The piston is excited by some external force $F_s(t)$, say, which could be exerted mechanically, via the support system, or produced by an acoustic source in the fluid. Piston motion disturbs the fluid to create an additional force $F(t)$ according to (30).

The characteristic equation for the piston is therefore

$$F(t) + F_s(t) + m\eta''(t) + \beta\eta'(t) + K\eta(t) = 0, \tag{32}$$

$$\{m_u\omega^2 - K_u + \beta_u i\omega\}\,\overline{\eta}(\omega) = \overline{F}_s(\omega). \tag{33}$$

$\overline{\eta}(\omega)$ and $\overline{F}_s(\omega)$ are the complex amplitudes of the piston motion and external force, respectively, and we have rearranged the terms to show how the flow has affected the apparent piston constants, the apparent mass, stiffness and damping factor in the presence of flow being given by

$$\left.\begin{aligned} m_u &= m + \tfrac{8}{3}\rho_0 a^3 - \tfrac{32}{45}\rho_0 a^3(\omega a/c)^2 + 4\rho_0 a^3 M^2, \\ K_u &= K - \rho_0 a U^2 A_0(1+\sigma M^2), \\ \beta_u &= \beta + \frac{\rho_0 a^4\omega^2\pi}{2c} + \frac{5\rho_0 a^4\omega^2\pi M^2}{3c} - \frac{\rho_0 a^4\omega^2\pi}{12c}\left(\frac{\omega a}{c}\right)^2. \end{aligned}\right\} \tag{34}$$

(A_0 and σ are constants to be determined by factors not incorporated in our analysis.)

This approximation is accurate to order M^2 and order $(ka)^2$, which are both small quantities. Equation (33) has no non-trivial real roots for $\overline{F}_s(\omega) = 0$, so that no self-sustained harmonic motions can exist. $\frac{8}{3}\rho_0 a^3$ is the well-known virtual mass of a baffled piston oscillating into a static fluid of density ρ_0 and $-\frac{32}{45}\rho a^3(\omega a/c)^2$ and $4\rho a^3 M^2$ are respectively the lowest-order compressible-fluid 'non-compactness' and Mach-number corrections to that term. These affect only the level of response, unlike the stiffness correction, which can easily result in instability and flutter. $\rho a^4\omega^2\pi/2c$ is the radiation damping, a term that, in accordance with our earlier analysis, is increased in flow by a factor

$$1 + \tfrac{10}{3}M^2 \tag{35}$$

over its static-fluid value.

The system is stable to small oscillations provided that $K_u > 0$. When this condition fails, the piston–spring system becomes unstable because the spring is not strong enough to counter the flow-induced suction force on the protruding piston; the equations then break down. This condition thus defines a critical parameter, α^2 say, which determines the stability of the system; it is stable provided that $\alpha^2 < 1$ but is otherwise unstable.

$$\alpha^2 = \rho_0 A_0 a U^2/K. \tag{36}$$

Provided that the system is stable, the resonance frequency is reduced by flow from its zero-velocity value

$$\Omega_0 = \left(\frac{K}{m+\frac{8}{3}\rho a^3}\right)^{\frac{1}{2}} \tag{37}$$

to

$$\Omega_u = \left(\frac{K-\rho a U^2 A}{m+\frac{8}{3}\rho a^3}\right)^{\frac{1}{2}} = \Omega_0(1-\alpha^2)^{\frac{1}{2}}. \tag{38}$$

Because of this reduced stiffness term flow always reduces the value of the resonance frequency. The resonance frequency is also reduced, but to a lesser extent, because flow increases the apparent mass.

When the piston is being irradiated above its no-flow resonance frequency the motion is dominated by the mass term, and flow will have little effect. However, if the forcing is below resonance there exists a flow velocity which brings the piston into resonance. Then the motion, and the sound field it generates, is enormously increased and is only controlled by damping.

Conversely, if the piston is initially set to resonate in the no-flow situation, any flow detunes the piston and causes a consequent reduction in radiated energy. In addition to these effects, the stability limit at $\alpha^2 = 1$ is probably the most significant. There always exists a flow speed above which the motion is unstable; flutter then occurs and the motion can no longer be described by linear equations.

5. Conclusions

We have shown that flow has two distinct influences on sound radiation from a compact vibrating surface. First, whenever linearized boundary conditions are appropriate flow over a source with constant volume displacement gives rise to a pressure increase proportional to $(1-M\cos\theta)^{-3}$. This cubic dependence on the Doppler factor arises from a convective change in source volume and from a quadratic dependence on the contracted wave scale, exactly as in aerodynamic sound problems. This implies an energy increase of $5M^2$ times the energy radiated into static fluid. The increased energy is drawn in the ratio 2 : 1 from the surface and mean flow respectively.

Even when the vibration induces an apparently monopole source whose amplitude is not altered by the flow, the convective amplification is evidently that appropriate to a quadrupole. This might in fact be expected, for it has long been recognized that aerodynamic sources are defined by the *rate* of change of the *rate* of mass addition and the double dependence on the time scale is quite characteristic of quadrupoles. Furthermore, the aerodynamic-noise analogy holds for the energy calculation, it being important to recognize that the sixth-power dependence on the Doppler factor for the energy field of a single source element is converted to a fifth-power dependence for a source region of fixed scale. This result contains a warning that the general integral formulae containing weaker convective effects on surface sources need very careful handling, and notice must be taken of the fact that simple source elements may be coupled to a degree that depends on the convection speed.

The amplitude of vibration is also altered, and there is a critical flow velocity above which local instability sets in. The flow changes the effective stiffness and alters the resonance frequency. These effects can be important, and arise principally from the flow-induced force that counters the action of the elasticity that tends to keep the surface in its equilibrium position.

The impedance of a baffled circular piston radiating into uniformly moving fluid was determined and the changes resulting from flow were shown in § 2 to be independent of the detailed piston geometry.

This work was conducted while D. J. Lovely was a research student in receipt of an S.R.C. grant. That support is gratefully acknowledged.

REFERENCES

BATCHELOR, G. K. 1967 *Introduction to Fluid Dynamics*. Cambridge University Press.

BENJAMIN, T. B. 1963 The threefold classification of unstable disturbances in flexible surfaces bounding inviscid flow. *J. Fluid Mech.* **16**, 436.

COPSON, E. T. 1947 On the problem of the electrified disk. *Proc. Edin. Math. Soc.* **3** (8), 14.

DEAN, P. 1972 Attenuation and *in situ* impedance measurements in flow ducts. *ISVR Flow Duct Acoustic Symp.*

FFOWCS WILLIAMS, J. E. 1963 The noise from turbulence convected at high speed. *Phil. Trans.* A **255**, 469.

FFOWCS WILLIAMS, J. E. 1974 Sound production at the edge of a steady flow. *J. Fluid Mech.* **66**, 791.

FFOWCS WILLIAMS, J. E. & HAWKINGS, D. L. 1969 Sound generated by turbulence and surfaces in arbitrary motion. *Phil. Trans.* A **264**, 321.

HOCH, R. & HAWKINS, R. 1974 Recent studies into Concorde noise reduction. *AGARD Conf. Proc.* no. 131.

LIGHTHILL, M. J. 1952 On sound generated aerodynamically. *Proc. Roy. Soc.* A **211**, 564.

LOWSON, M. V. 1965 The sound field of singularities in motion. *Proc. Roy. Soc.* A **286**, 559.

MANI, R. 1974 The jet density exponent issue for the noise of heated subsonic jets. *J. Fluid Mech.* **63**, 611.

TESTER, B. 1973 The propagation and attenuation of sound in lined ducts containing uniform or plug flow. *J. Sound Vib.* **28**, 151.